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LETTER TO THE EDITOR

Complex-temperature singularities and their application to the thermodynamic description of multifractals

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Abstract. It is shown that applicability of standard high-temperature expansions in multifractal thermodynamics is restricted by complex-temperature singularities. An analytic continuation method (finite-temperature expansions) has been developed to improve the analytic expansions approach. The quantum intermittency is then used as an example of applicability of the finite-temperature expansion.

It is well known that linear approximation of the generalized fractal dimensions D_q (or parabolic approximation of the $f(\alpha)$ distribution) is generally applicable in a narrow vicinity of $q = 0$ only (see, for instance [1, 2] and references therein). On the other hand, a quantitative description of multifractals beyond the linear approximation is now actually due to the possibility of obtaining rather accurate numerical data for different physical systems. In particular, quantum intermittency has recently [3–9] been intensively studied beyond the linear approximation. In this letter we will show that the concept of analytical continuation of standard thermodynamic quantities on a complex plane can be adapted in the context of multifractal measures. This adaptation should then provide an improvement of the standard linear approximation of the generalized dimensions D_q. We then apply this finite-temperature (FT) approximation (formulae (14) and (21)) to a model (Baker transformation) where D_q are known analytically and where the complete analytical continuation can be calculated and shown to be better than the high-temperature expansion (formula (10)) in a certain range of values of q (figure 1). We also use results of recent numerical simulations [7, 8] of the quantum intermittency as an example of applicability of the FT-approximation to real physical systems.

For quantum systems the generalized dimensions spectrum can be defined as follows [7]

$$D_q = \lim_{l\to 0} \frac{\ln Z(q)}{(q-1)\ln l} \tag{1}$$

where the partition function

$$Z(q) = \sum_i \mu_i^q \tag{2}$$

and the interval $[0, 2\pi]$ is partitioned into small intervals of size l, the ith of which receives a weight μ_i from the spectral measure (the spectrum in that case is the whole unit circle). This means that in the limit $l \to 0$ the partition function $Z(q,l)$ behaves as a power law

$$Z(q) \sim l^{\tau(q)}$$

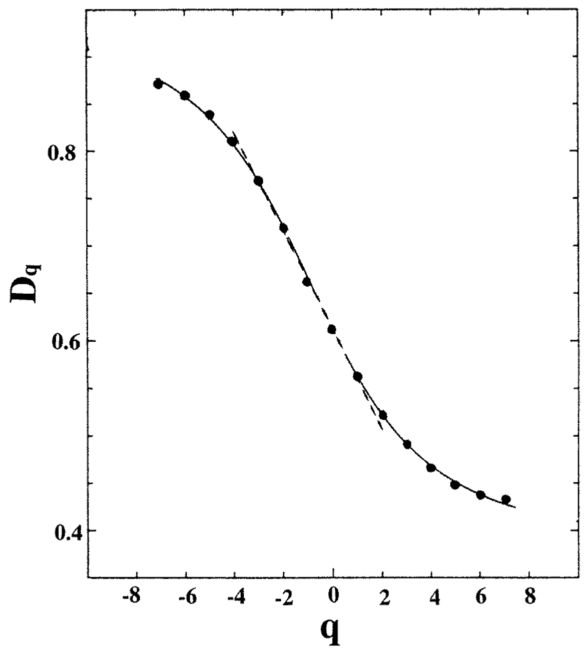

Figure 1. Generalized dimensions D_q against q for a strange attractor of the Baker map. The broken *straight* line is drawn for comparison with the high-temperature (linear) approximation (10), the full curve corresponds to the first-order approximation of the FT expansion (21) and the dots correspond to analytical computations.

where

$$\tau(q) = D_q(q-1). \tag{3}$$

On the other hand the partition function can be represented as follows [10]

$$Z(q) \simeq \int \rho(\alpha) l^{q\alpha - f(\alpha)}\, \mathrm{d}\alpha \tag{4}$$

where α represents the singularity strength of the measure μ and $f(\alpha)$-singularity spectrum describes the statistical distribution of the singularity exponent α. If we cover the support of the measure μ with balls of size l, the number of such balls that scale like l^{α}, for a given α, behaves like $N_\alpha(l) \sim l^{-f(\alpha)}$. In the limit $l \to 0$, the sum (4) is dominated by the term $l^{\min_\alpha(q\alpha - f(\alpha))}$. Then from the definition of $\tau(q)$, one obtains

$$\tau(q) = \min_\alpha(q\alpha - f(\alpha)). \tag{5}$$

Thus, the $\tau(q)$ is obtained by Legendre transforming the $f(\alpha)$. When $f(\alpha)$ and $\tau(q)$ are smooth functions, relationship (5) can be rewritten in the following way

$$\tau(q) = q\alpha - f(\alpha) \qquad \frac{\mathrm{d}f}{\mathrm{d}\alpha} = q. \tag{6}$$

The thermodynamics interpretation of these relationships means that q can be interpreted as an inverse temperature $q = T^{-1}$ and the limit $l \to 0$ can be seen as the thermodynamic limit of infinite volume ($V = \ln 1/l \to \infty$). Then by identifying $\alpha_i = \ln \mu_i / \ln(1/l)$ to the energy E_i (per unit of volume) of a microstate i, one can rewrite the partition function under the familiar form

$$Z(q) = \sum_i \exp(-qE_i). \tag{7}$$

From the definition: $f(\alpha) = \ln N_\alpha(l)/\ln(1/l)$, the singularity spectrum $f(\alpha)$ plays the role of the entropy (per unit of volume).

Expansion of the entropy $f(\alpha(q))$ in power series (the high-temperature expansion)

$$f(q) = f(0) + q\left(\frac{\mathrm{d}f}{\mathrm{d}q}\right)_{q=0} + q^2\frac{1}{2}\left(\frac{\mathrm{d}^2 f}{\mathrm{d}q^2}\right)_{q=0} + \cdots \tag{8}$$

in the first-order approximation is

$$f(q) \simeq f(0) + q^2\frac{1}{2}\left(\frac{\mathrm{d}^2 f}{\mathrm{d}q^2}\right) \tag{9}$$

since generally $(\mathrm{d}f/\mathrm{d}q)_{q=0} = 0$ (see (6)).

The generalized dimensions spectrum corresponding to (9) is (see, for instance, [4] and references therein)

$$D_q \simeq D_0 + aq \tag{10}$$

where a is some constant.

It is known that entropy can have singularities in the complex temperature plane (see, for instance [11–13] and references therein). If the multifractal entropy $f(q)$ has singularities on the complex q-plane, then the radius of convergence of the *real* Maclaurin series expansion (8) is determined by the distance from the point $q = 0$ to a nearest singularity of $f(q)$ on the complex plane. One could then use the standard procedure of analytic continuation to obtain power series expansions beyond the circle of convergence of the expansion (8)

$$f(q) = f(\pm|q_0|) + (q \mp |q_0|)\left(\frac{\mathrm{d}f}{\mathrm{d}q}\right)_{q=\pm|q_0|} + (q \mp |q_0|)^2\frac{1}{2}\left(\frac{\mathrm{d}^2 f}{\mathrm{d}q^2}\right)_{q=\pm|q_0|} + \cdots \tag{11}$$

where $|q_0|$ is the modulus of the nearest to point $q = 0$ complex-temperature singularity and we introduce indexes $(\pm)$ to distinguish between case $q > 0 - (+)$ and case $q < 0 - (-)$.

Let us rewrite (11) in a form similar to (8)

$$f(q) = A^{(\pm)} + B^{(\pm)}q + \cdots \tag{12}$$

where

$$A^{(\pm)} = f(\pm|q_0|) \mp |q_0|\left(\frac{\mathrm{d}f}{\mathrm{d}q}\right)_{q=\pm|q_0|} \qquad B^{(\pm)} = \left(\frac{\mathrm{d}f}{\mathrm{d}q}\right)_{q=\pm|q_0|}. \tag{13}$$

Using (6) it is easy to show that $\tau(q)$ corresponding to (12) has the following form

$$\tau_q^{(\pm)} = -A^{(\pm)} + (C^{(\pm)} - B^{(\pm)})q + B^{(\pm)}q\ln|q| + \cdots \tag{14}$$

where $C^{(\pm)}$ are some constants. One can see that an additional 'logorithmic' term appears in the FT-expansion (14). An *a priori* reason for the 'logorithmic' term could be related to so-called dimension invariance [14].

Along the circle $|q| = 1$ (on the complex plane) this 'logarithmic' term is absent and the FT-expansion (14) takes a Maclaurin-like form

$$\tau_q^{(\pm)} = -A^{(\pm)} + (C^{(\pm)} - B^{(\pm)})q + \cdots. \tag{15}$$

One can use this formal property to simplify the FT-expansion (14). Indeed, if: (a) there exists a segment of the circle $|q| = 1$ which belongs to both $(+)$ and $(-)$ circles of convergence and (b) τ_q is fixed by the first two terms of expansions (15) with sufficient accuracy along this segment, then the two expansions (15) ((+) and (−) cases) can be considered as Maclaurin-like expansions of the *same* analytic function. Therefore

$$A^{(+)} = A^{(-)} = A \tag{16}$$

$$C^{(+)} - B^{(+)} = C^{(-)} - B^{(-)}. \tag{17}$$

Using the condition $\tau(1) = 0$ (see (3)) we obtain from (15)

$$A^{(+)} = C^{(+)} - B^{(+)} \tag{18}$$

and then using (16)–(18) we can rewrite (14) as follows

$$\tau_q = A(q-1) + B^{(\pm)} q \ln|q| + \cdots. \tag{19}$$

If such a segment of circle $|q| = 1$ does not exist for a considered multifractal, then one should use the general form, (14), of the FT-expansion. As we shall show below, the data obtained in numerical simulations of some interesting physical systems are well fitted by the simplified form (19).

It follows from (3) and (19) that

$$D_q = A + B^{(\pm)} \frac{q \ln|q|}{(q-1)} + \cdots. \tag{20}$$

Using (20) at $q = -1$ we obtain $A = D_{-1}$, and then finally

$$D_q \simeq D_{-1} + B^{(\pm)} \frac{q \ln|q|}{(q-1)}. \tag{21}$$

Approximation (21) is obviously wrong in a narrow vicinity of the point $q = 0$ since it gives $D_0 = D_{-1}$, which is almost never the case for a multifractal. Thus, in the case when point $q = 0$ belongs to the circles of convergence of the FT-expansions one should take into account more than the first two terms of the FT-expansions (11) to obtain an accurate representation of $f(q)$ (and, consequently, D_q) in the narrow vicinity of the point $q = 0$. Below we shall show that outside the narrow vicinity of the point $q = 0$ these first two terms can give a good approximation of the generalized dimensions D_q in some representative interval of the inverse temperature q (both for $q > 0$ and $q < 0$).

To show this let us start from the multifractality of a strange attractor of the Baker map for which analytical results are available. This transformation is defined as

$$\begin{aligned} &[x_{n+1}, y_{n+1}] = [l_1 x_n, y_n/\eta] \qquad y_n < \eta \\ &[x_{n+1}, y_{n+1}] = [\tfrac{1}{2} + l_2 x_n, (y_n - \eta)/(1-\eta)] \qquad y_n > \eta. \end{aligned} \tag{22}$$

The attractor of this map consists of an infinite number of lines in the y direction which intersect a horizontal line in two interwoven Cantor sets. These sets are characterized by contraction rates l_1 and l_2, and are visited with probability η and $1 - \eta$, respectively. The dimension spectrum D_q of the cross section follows from

$$\frac{\eta^q}{l_1^{(q-1)D_q}} + \frac{(1-\eta)^q}{l_2^{(q-1)D_q}} = 1. \tag{23}$$

If we introduce definitions $\eta^q = a$, $(1-\eta)^q = b$, $l_2^{(q-1)D_q} = G$ and $\ln l_1 / \ln l_2 = k$, then we can rewrite (23) as

$$G^k - bG^{(k-1)} - a = 0. \tag{24}$$

From this eqution one obtains

$$\frac{\mathrm{d}G}{\mathrm{d}q} = \frac{\mathrm{d}a/\mathrm{d}q + (\mathrm{d}b/\mathrm{d}q)G^{(k-1)}}{kG^{(k-1)} - b(k-1)G^{(k-2)}}.$$

$\mathrm{d}G/\mathrm{d}q$ has a singularity when $G = b(k-1)/k$. Substituting this relationship into (24) we obtain values of q for which $\mathrm{d}G/\mathrm{d}q$ is singular

$$q_0 = \frac{\ln c}{\ln[\eta/(1-\eta)^k]} \tag{25}$$

where

$$c = -\frac{(k-1)^{(k-1)}}{k^k}. \tag{26}$$

Since

$$f(q) = D_q + q(q-1)\frac{\mathrm{d}D_q}{\mathrm{d}q} \tag{27}$$

entropy $f(q)$ of the Baker map also has singularities at the same values of $q = q_0$.

The constant c is positive when $(k-1) = -1/n$, where $n = 3, 5, 7 \ldots$. For these specific values of k corresponding values of q_0 are real numbers and, $\mathrm{d}G/\mathrm{d}q$ (and, consequently, $f(q)$) has singularities on the real axis. In the general case, however, the values of q_0 are complex.

Let us consider an example with concrete values of $\eta = 0.6$, $l_1 = 0.25$ and $l_2 = 0.4$. For this case we obtain from (25) and (26) the complex value of $q_0 \simeq -1.1 + \mathrm{i}3.6$ and, consequently $|q_0| \simeq 3.8$. This is the radius of convergence of the high-temperature expansion (8) for the Baker map. Corresponding radii of convergence of the finite-temperature expansions (11) are $R^{(+)} \simeq 5.8$ and $R^{(-)} \simeq 4.1$, so that the interval of applicability of the FT-expansion is approximately $-8 < q < 10$. For the first-order approximation we should exclude a narrow vicinity of the point $q = 0$ from this interval. Figure 1 shows a set of values D_q for this situation. The broken straight line in this figure corresponds to the high-temperature expansion (10) whereas the full curve corresponds to the FT-expansion (21) (dots correspond to analytical results). One can see good agreement with approximation (21) for $0.5 < q < 7$ and $-7 < q < -0.5$.

Let us apply this approach to some quantum systems with multifractal energy spectra. Figure 2 (adapted from [7]) shows the generalized dimensions spectrum D_q for multifractal data (dots) obtained in a recent numerical simulation of a quasiperiodically driven spin-$\frac{1}{2}$ system with a singular continuous spectrum. In this figure we show D_q against $q\ln|q|/(q-1)$ and D_q against q (the latter case is shown in the inset figure). Full lines in both of these figures are drawn for comparison with FT-approximation (21). In the main figure the upper *straight* line indicates agreement with (21) for $q < 0$ whereas the lower straight line indicates agreement with (21) for $q > 0$. In the inset figure the broken straight line is drawn for comparison with the high-temperature approximation (10) (the full curve in the inset figure corresponds to (21)).

To apply this approach to the quantum pseudo-diffusion let us recall some definitions. If the wavepacket is initially localized with $\psi_n = \delta_{n0}$ at time $t = 0$, the pth time-dependent displacement moment of the wavepacket might be expected to grow as

$$\langle r^q\rangle = \Sigma_n |n|^q |\psi_n(t)|^2 \sim t^{q\beta(q)}$$

where n measures the displacement. If $\beta(p)$ does not equal a constant then this generalized diffusion law corresponds to intermittent behaviour of the quantum systems [6–8]. In [8] relationship

$$\beta(q) = D_{1-q} \tag{28}$$

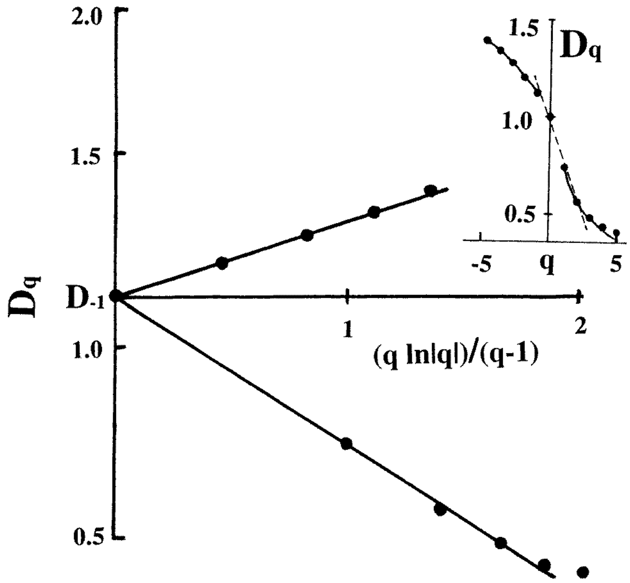

Figure 2. Generalized dimensions D_q for a quasiperiodically driven spin-$\frac{1}{2}$ system with singular continuous spectrum. The dots are results of numerical simulation performed in [7] and full lines are drawn for comparison with the FT-approximation (21) (see text for more detail).

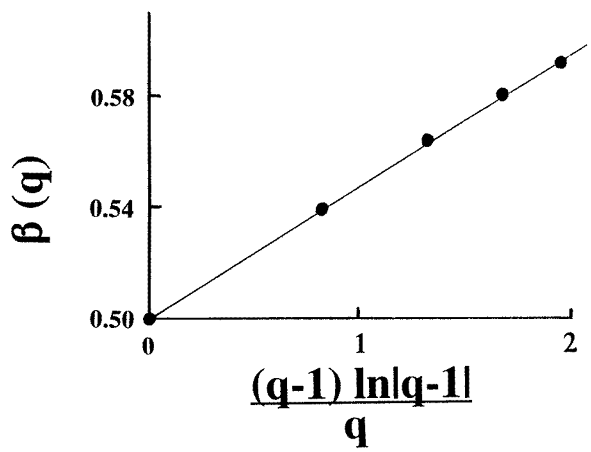

Figure 3. Pseudo-diffusion exponent $\beta(q)$ against $\frac{(q-1)\ln|1-q|}{q}$ for $q = 2, \ldots, 10$. The data (dots) are taken from [8] for irrational $\omega = [0, 4]$. The full straight line is drawn for comparison with (29).

is suggested for the quasiperiodic quantum systems with singular continuous spectrum. If we substitute D_q given by (21) into (28) we obtain

$$\beta(q) \simeq D_{-1} + B^{(-)} \frac{(q-1)\ln|1-q|}{q} \tag{29}$$

for $q > 2$ (it should be noted that $q > 2$ in (29) corresponds to $q < -1$ in (21)). The author of [8] supports relationship (28) by numerical results obtained for the Harper model in the critical regime. This is a Schrödinger equation with hopping amplitude $t_{i,i+1} = -1$

and on-site potential $v_i = 2\cos(2\pi\omega\mathrm{i} + \phi)$ where ω is an irrational and ϕ is an arbitrary phase. For this model, the exponent $\beta(q)$ was computed in [8] using wavepackets evolution. Results of these calculations (dots) are shown in figure 3 (adapted from [8] for $\omega = [0, 4]$). The axes in this figure are chosen for comparison with FT-approximation (29). One can see good agreement between data (dots) and FT-approximation (29) (straight line).

We use these numerical results as an illustrative example only. Therefore, we do not investigate them in detail. On the other hand, the method developed in this letter seems independent of the physical systems where the multifractal appears. It appears that this method could therefore be applied in principle to a wide range of physical systems. Obtaining limits of validity of this method is an interesting open problem for future investigations.

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